A Local Existence and Uniqueness Theorem for First Order Differential Equations

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1 Introduction

Differential equations are wildly popular in modeling, especially in time-dependent processes, and are a staple in math courses after introductory calculus (unless you go to Harvard whoops). We're typically interested in solving them, especially when given some initial conditions, to explicitly understand the behavior of our system of interest. In this pursuit, two fundamental questions arise:

- Does a solution exist?
- If so, how many solutions are there?

In this paper, we'll address these questions by proving an **existence and uniqueness theorem** under simple, specific circumstances.

1.1 A Chemical Kinetics Example

Before we get into that theorem, though, we'll introduce a simple applied example that we will consult with several times throughout the paper to see why the theorem is useful, and what it can show us.
In chemistry and biochemistry, we often have reactions of the form

$$A \to B,$$

where A and B are some chemical compounds/enzymes. In some situations, we may propose a first-order reaction rate; that is, we model that the speed of the reaction is proportional to the concentration of A. Suppose we empirically measure that proportion to be -0.2; then mathematically, we can get the following differential equation:

$$\frac{dA}{dt} = -0.2A. \tag{1}$$

Note that the sign is negative since A is being *consumed* through the course of the reaction.

This type of equation can be solved analytically, and is one of the first that you would learn about in a typical differential equations class. Since the derivative of A is proportional to A, this reminds us of an exponential function (since $\frac{d}{dt}e^{kt} = ke^{kt}$), so we could propose the following general solution for A as a function of time (for any $C \in \mathbb{R}$):

$$A(t) = e^{-0.2t} + C. \tag{2}$$

Suppose that we observe the concentration of A at time 0 to be 1 (even though the reaction may proceed for some time before or after). Then we also have an **initial value**:

$$A(0) = 1. \tag{3}$$

The system of describes created by equations 1 and 3 is an example of an **initial value problem (IVP)**; we will derive a theorem for the solutions of this class of IVP during this paper. For this specific IVP, thought, we could set the constant in the general solution, equation 2, to $C = 0$ to solve for the specific solution

$$A(t) = e^{-0.2t}. \tag{4}$$

Note that this is a solution to the IVP since

$$A'(t) = -0.2e^{-0.2t} = -0.2A(t),$$

which satisfies 1, and also

$$A(0) = e^{-0.2\times 0} = 1,$$

which satisfies 3.

However, we were lucky to have a problem that we happened to have a simple solution to - we don't know if this is the only solution, or if any solution even has to exist in this type of problem. Answering those questions is the goal of an existence and uniqueness theorem, which we'll now introduce.

1.2 Statement of the Existence and Uniqueness Theorem [1, 2]

Consider the general first-order scalar initial value problem (**IVP**):

$$y'(x) = F(x, y(x)) \tag{5}$$

$$y(x_0) = y_0, \tag{6}$$

where F defined on the rectangle $R = [x_0 - a, x_0 + a] \times [y_0 - b, y_b + b]$, with F and $\frac{\partial F}{\partial y}$ both continuous on R.

Since F is defined on a closed interval and F and $\frac{\partial F}{\partial y}$ are continuous, both are bounded on R; thus, there exist constants $M, K > 0$ such that

$$|F(x, y)| \leq M, \tag{7}$$

$$\left|\frac{\partial F}{\partial y}(x, y)\right| \leq K. \tag{8}$$

Existence and Uniqueness Theorem. *Let* $\alpha = \min\{a, b\}$*. For any* $h \leq \min\left\{\alpha, \frac{\alpha}{M+K\alpha}\right\}$*, there exists a unique* $\mathcal{C}^1$ *function defined on* $B_h[x_0]$ *that solves the IVP described by equations 5 and 6.*

1.2.1 Notation Notes

We'll also introduce some frequently used notation/basic definitions here:

Definition. *We will use closed intervals often, so we introduce the concept of a* ***closed ball:*** $B_r[z] = [z-r, z+r]$.

- Thus, we could alternatively define $R = B_a[x_0] \times B_b[y_0]$.

Definition. *The* ***supremum****, or* ***least upper bound****, of a function f on a set S is the smallest Y such that for all $x \in S$, $f(x) \leq Y$.*

Consider some examples:

- For the function $f(x) = x^2$ on set $S = \{0, 1, 2\}$, since the image of S has a maximum, we just have the supremum being 4.
- However, the supremum of $f(x) = x^2$ on the set $S = (0, 1)$, where the image of S is also $(0, 1)$ and thus doesn't contain a maximum. The supremum is now 1, which is the smallest number larger than all of the values in the image of S.

Definition. *Let $g(x) = |f(x)|$ give the absolute value of f for the functions f, g with domain S. Then $\|f\|_S$ denotes the supremum of the image of g, which is the supremum of the absolute values of f's image.*

Thus, we could show our bounds for the problem as

$$\|F\|_R \leq M$$
$$\left\|\frac{\partial F}{\partial y}\right\|_R \leq K.$$

1.2.2 Use of the Theorem for the Kinetics Example

For our kinetics example, we already stated our initial value problem:

$$A'(t) = -0.2A = F_1(t, A)$$
$$A(0) = 1.$$

To fit the theorem, we need to define F_1 on some rectangle for t and A; this means that we need bounds for both variables. Suppose we're mainly interested in the reaction's progress in the seconds before and after $t = 0$; then we can say t must be in the closed ball $B_1[0]$. For the bound of A, we can then cheat a bit (since we know a solution already) to see that we could have $A(-1) = e^{0.2}$ and $A(1) = e^{-0.2}$ by our intuitively derived solution in equation 4. A ball that contains those bounds is $B_{e^{0.2}}[1]$. We can thus define F_1 on the rectangle $R_1 = B_1[0] \times B_{e^{0.2}}[1]$.

Note that $F_1(t, A) = -0.2A$ is of class $\mathcal{C}^\infty$, so both F_1 and $\frac{\partial F_1}{\partial A}$ are continuous on R_1. Additionally, the maximum absolute value that F_1 takes on R_1 is clearly $0.2 \times e^{0.2}$, so

$$\|F_1\|_{R_1} \leq 0.2e^{0.2}.$$

Also, since $\frac{\partial F_1}{\partial A} = -0.2$,

$$\left\|\frac{\partial F_1}{\partial A}\right\|_{R_1} \leq 0.2.$$

Then, we let $\alpha_1 = \min\{1, e^{0.2}\} = 1$. The existence and uniqueness theorem states that for any $h_1 \leq \min\{\alpha_1, \frac{\alpha_1}{0.2e^{0.2}+0.2\alpha_1}\} = 1$, there exists a unique $\mathcal{C}^1$ function $A(t)$ defined on $B_1[0]$ that satisfies the IVP.
Since we already found a solution, by the existence and uniqueness theorem, we know that our previously derived solution $A(t) = e^{-0.2t}$ is the one and only solution for the IVP with t and A in the rectangle $R_1 = B_1[0] \times B_{e^{0.2}}[1]$.

1.3 Roadmap

For most of this paper, we will be deriving the existence and uniqueness theorem. This proof is quite lengthy (spoiler alert: it's the whole project), so we will outline the direction of the proof here:

1. We will establish the equivalence of the IVP to an integral equation. The existence and uniqueness theorem will be justified on the integral form. Since the IVP and integral equation are equivalent, those results proved for the integral form will hold for the IVP.

2. The *uniqueness* of a solution will be shown. This can be done through a strategy we have frequently used in injectivity arguments in Math 22 - we will assume that there are two solutions, then show that they are the same.

3. The proof of the *existence* of a solution will be centered around the process of **Picard iteration**. This method starts with an estimate for the solution function, $\Phi_0(x) = y_0$, and generates a recursive sequence, where each term is a function of the previous one: $\Phi_n(x) = h(\Phi_{n-1}(x))$. With the specific choice of h that we will use, the terms become better and better approximations for the solution, and the sequence will converge nicely to a function Φ, the true solution function to the IVP. This section will consist of several steps:

   (a) We will first show that such a sequence of Φ_j stays within the constraints of the problem - that is, the graph of each estimate remains in our constraint rectangle R; symbolically, $graph(\Phi_j) \subset R$. This will ensure that the sequence converges to a valid solution of the problem, and is just a basic sanity check.

   (b) We'll then talk about **uniform convergence**. This is a stronger form of convergence that will allow us to show that the whole *function* Φ_n converges to Φ as $n \to \infty$, as opposed to just the pointwise convergence $\Phi_n(x) \to \Phi(x)$ as $n \to \infty$. This will be important in helping us show that the limit function Φ is actually a solution to the IVP. We'll also discuss **uniformly Cauchy sequences**, which allow us to more easily show that a recursive sequence is uniformly convergent, without actually knowing the limit Φ.

   (c) Finally, we'll apply these results to show that the sequence converges to the function Φ. We'll show that this is limit function is actually the solution to the IVP, and is of class $\mathcal{C}^1$.

2 Equivalence of the IVP to an Integral Equation [1]

In this section, we simply want to express the IVP as an integral instead of a derivative, which may seem more complicated, but will provide a path for the rest of the proof. Since the two forms will be equivalent, any result proved about the integral equation holds for the IVP, including the whole existence and uniqueness theorem.

Lemma 1. *Let $\psi : B_h[x_0] \to B_b[y_0]$ be a continuous function. Then ψ solves the IVP*

$$\psi'(x) = F(x, \psi(x)) \tag{9}$$

$$\psi(x_0) = y_0 \tag{10}$$

if and only if it satisfies the integral equation

$$\psi(x) = y_0 + \int_{x_0}^{x} F(t, \psi(t))dt. \tag{11}$$

Proof. ($\Rightarrow$) Suppose ψ satisfies the IVP. Since ψ is continuous, we can integrate both sides of equation 9 using the fundamental theorem of calculus to get

$$\int_{x_0}^{x} \psi'(t)dt = \int_{x_0}^{x} F(t, \psi(t))dt$$

$$\psi(x) - \psi(x_0) = \int_{x_0}^{x} F(t, \psi(t))dt$$

$$\psi(x) = \psi(x_0) + \int_{x_0}^{x} F(t, \psi(t))dt.$$

Plugging in equation 10, we get

$$\psi(x) = y_0 + \int_{x_0}^{x} F(t, \psi(t))dt,$$

which is exactly equation 11.

($\Leftarrow$) Suppose ψ satisfies equation 11. Since the integral in equation 11 is differentiable, then ψ must also is differentiable, so we can simply differentiate both sides of equation 11 with respect to x, again utilizing the fundamental theorem of calculus to get

$$\frac{d}{dx}\psi(x) = \frac{d}{dx}\left[\psi(x_0) + \int_{x_0}^{x} F(t, \psi(t))dt\right]$$

$$\psi'(x) = 0 + \frac{d}{dx}\int_{x_0}^{x} F(t, \psi(t))dt$$

$$\psi'(x) = F(x, \psi(x)),$$

which matches equation 9 exactly. We can also show that the initial value, equation 10, is satisfied by plugging in $x = x_0$ into the integral equation, equation 11:

$$\psi(x_0) = y_0 + \int_{x_0}^{x_0} F(t, \psi(t))dt = y_0 + 0 = y_0.$$

Thus, we have shown that the initial value problem described in equations 9 and 10 is equivalent to the integral equation 11. □

2.1 Integral Equation for the Kinetics Example

We have the following IVP from our kinetics example:

$$\psi_1'(t) = -0.2\psi_1(t)$$
$$\psi_1(0) = 1.$$

By the equivalence that we just proved, some continuous function $\psi_1(t)$ is a solution to the IVP if and only if it satisfies the following integral equation:

$$\psi_1(t) = 1 + \int_0^t -0.2\psi_1(t)dt. \tag{12}$$

3 Uniqueness of a Solution [1]

We will now show the uniqueness of solutions to the integral equation by considering two solutions, Φ and Ψ, and showing that they are equal. We will show this for $x > x_0$ (while still being in $B_h[x_0]$), and the proof for $x < x_0$ is analogous.

Proof. If each of Φ and Ψ are solutions to the integral equation, then by definition we have

$$\Phi(x) = y_0 + \int_{x_0}^x F(t, \Phi(t))dt \tag{13}$$
$$\Psi(x) = y_0 + \int_{x_0}^x F(t, \Psi(t))dt. \tag{14}$$

To relate these equations, we can subtract equation 14 from equation 13:

$$\Phi(x) - \Psi(x) = \int_{x_0}^x [F(t, \Phi(t)) - F(t, \Psi(t))]\, dt.$$

$$|\Phi(x) - \Psi(x)| = \left| \int_{x_0}^x [F(t, \Phi(t)) - F(t, \Psi(t))]\, dt \right| \tag{15}$$

Now, consider that $|F(t, \Phi(t)) - F(t, \Psi(t))| \geq F(t, \Phi(t)) - F(t, \Psi(t))$ for all t, so

$$\left| \int_{x_0}^x [F(t, \Phi(t)) - F(t, \Psi(t))]\, dt \right| \leq \left| \int_{x_0}^x |F(t, \Phi(t)) - F(t, \Psi(t))| dt \right|.$$

Then, since $x > x_0$, the integral on the right-hand side is positive, so we have

$$\left| \int_{x_0}^x |F(t, \Phi(t)) - F(t, \Psi(t))| dt \right| = \int_{x_0}^x |F(t, \Phi(t)) - F(t, \Psi(t))| dt$$

Plugging into equation 15, we get

$$|\Phi(x) - \Psi(x)| \leq \int_{x_0}^x |F(t, \Phi(t)) - F(t, \Psi(t))| dt. \tag{16}$$

A key insight here is that we can recall $\frac{\partial F}{\partial y}$ is continuous and bounded. With that condition, we can apply the Mean Value Theorem for that partial derivative. Specifically, we know that there exists some c in between $\Phi(t)$ and $\Psi(t)$ such that

$$F(t,\Phi(t)) - F(t,\Psi(t)) = \frac{\partial}{\partial y}F(t,c)(\Phi(t)-\Psi(t)).$$

By that aforementioned bound (equation 8), $\frac{\partial F}{\partial y} \le K$. Plugging this in,

$$F(t,\Phi(t)) - F(t,\Psi(t)) \le K(\Phi(t)-\Psi(t)).$$

With K being non-negative, we can plug this into equation 16 to get

$$|\Phi(x)-\Psi(x)| \le K\int_{x_0}^{x} |\Phi(t)-\Psi(t)|dt \tag{17}$$

This is now a very convenient form, which can be made obvious through a variable change. Let

$$\mathcal{U}(x) = \int_{x_0}^{x} |\Phi(t)-\Psi(t)|dt. \tag{18}$$

From here, we want to show that $\mathcal{U}(x) = 0$, since that would show that $\Phi(t) = \Psi(t)$ for all $t \in [x_0, x_0+h]$.
Note that since $x > x_0$, as the integral of a nonnegative function

$$\mathcal{U}(x) \ge 0. \tag{19}$$

Thus, we next want to show that $\mathcal{U}(x) \le 0$ to complete the goal. We can substitute $\mathcal{U}(x)$ into equation 17 to become

$$\mathcal{U}'(x) \le K\mathcal{U}(x).$$

This may remind you of an exponential function - consider that for some $f(x) = e^{kx}$, we have $f'(x) = ke^{kx} = kf(x)$. We can make this appear by multiplying by $e^{-K(x-x_0)}$:

$$\begin{aligned}\mathcal{U}'(x) &\le K\mathcal{U}(x)\\ \mathcal{U}'(x) - K\mathcal{U}(x) &\le 0\\ \mathcal{U}'(x)e^{-K(x-x_0)} - K\mathcal{U}(x)e^{-K(x-x_0)} &\le 0.\end{aligned}$$

This is now the aftermath of a product rule derivative, so we can condense to see

$$\mathcal{U}'(x)e^{-K(x-x_0)} - K\mathcal{U}(x)e^{-K(x-x_0)} = \frac{d}{dx}\left[\mathcal{U}(x)e^{-K(x-x_0)}\right] \le 0.$$

We can now integrate both sides of this inequality in order to isolate $\mathcal{U}(x)$ terms:

$$\begin{aligned}\int_{x_0}^{x} \frac{d}{dt}\left[\mathcal{U}(t)e^{-K(t-x_0)}\right]dt &\le \int_{x_0}^{x} 0dt\\ \mathcal{U}(x)e^{-K(x-x_0)} - \mathcal{U}(x_0)e^{-K(x_0-x_0)} &\le 0.\end{aligned}$$

Note that by definition, $\mathcal{U}(x_0) = \int_{x_0}^{x_0} |\Phi(t) - \Psi(t)|dt = 0$, so this becomes

$$\mathcal{U}(x)e^{-K(x-x_0)} \leq 0.$$

As an exponential, $e^{-K(x-x_0)} > 0$, so this means

$$\mathcal{U}(x) \leq 0. \tag{20}$$

Thus, combining equations 19 and 20, we have that $\mathcal{U}(x) \geq 0$ and $\mathcal{U}(x) \leq 0$, so it must be that $\mathcal{U}(x) = 0$. Recalling the definition of $\mathcal{U}$ again (equation 18), this means

$$\int_{x_0}^{x} |\Phi(t) - \Psi(t)|dt = 0.$$

As an absolute value, the integrand is always non-negative. Observe that $\Phi(t)-\Psi(t)$ must be continuous (as the difference of continuous functions), so $|\Phi(t) - \Psi(t)|$ must be continuous as well. For contradiction's sake suppose that $|\Phi(t_1) - \Psi(t_1)| > 0$ for some valid t_1. Then there must exist some $\epsilon > 0$ such that $|\Phi(t) - \Psi(t)| > 0$ for all $t \in [t_1 - \epsilon, t_1 + \epsilon]$. However, then we could split up the integral to find

$$\int_{x_0}^{x} |\Phi(t) - \Psi(t)|dt \geq \int_{t_1-\epsilon}^{t_1+\epsilon} |\Phi(t) - \Psi(t)|dt > 0,$$

which is a contradiction. Therefore, we must have that $\Phi(t) = \Psi(t)$ for all $x_0 < t < x_0 + h$; as previously mentioned, a similar proof follows for $t < x_0$.
With that result, we would have that $\Phi(t) = \Psi(t)$ for all $t \in B_h[x_0]$, so they are the same solution. It follows that if the integral equation has a continuous solution, it is a unique solution - thus, if the initial value problem has a continuous solution, it is a unique solution. □

4 Existence: Introduction to Picard Iteration [1]

So far, we have shown that if a continuous solution to the IVP exists, it is the unique continuous solution. Recall the general IVP:

$$y'(x) = F(x, y(x))$$
$$y(x_0) = y_0,$$

where F defined on the rectangle $R = B_a[x_0] \times B_b[y_0]$, with F and $\frac{\partial F}{\partial y}$ both continuous on R.

We now want to show that a solution must actually exist. In combination with the previous section, this will show that there is a single, unique solution. We will construct this solution through **Picard iteration**, where we recursively find better and better solution approximations - where each approximation is a function - in the following fashion:

$$
\begin{aligned}
\Phi_0(x) &= y_0 \\
\Phi_1(x) &= y_0 + \int_{x_0}^{x} F(t, \Phi_0(t))dt \\
&\vdots \\
\Phi_n(x) &= y_0 + \int_{x_0}^{x} F(t, \Phi_{n-1}(t))dt. \\
&\vdots
\end{aligned}
$$

In doing this, we want this sequence to converge to a function Φ that is the actual solution to the problem. In order for Φ to be valid, though, we must have that $(x, \Phi_n(x)) \in R$ for all $x \in B_a[x_0]$; since R is closed and thus contains its boundaries, we can reasonably expect that the limit of the infinite sequence will also be contained in R. We show this in our first lemma.

Lemma 2. *Let g be a continuous function on $B_h[x_0]$, and define*

$$h(g(x)) = y_0 + \int_{x_0}^{x} F(t, g(t))dt.$$

If g satisfies $|g(x) - y_0| \leq b$ for all $x \in B_h[x_0]$, then $h \circ g$ also satisfies

$$|h(g(x)) - y_0| \leq b.$$

Proof. We will again show for $x \geq x_0$, with an analogous proof existing for $x < x_0$. Suppose that $|g(x) - y_0| \leq b$ for all valid $x \in B_h[x_0]$. We'll first evaluate $|h(g(x)) - y_0|$ by direct substitution:

$$|h(g(x)) - y_0| = \left| \int_{x_0}^{x} F(t, g(t)) \right| dt \leq \int_{x_0}^{x} |F(t, g(t))| dt.$$

Recall that $F(x, y) \leq M$ in R (where M is non-negative), which gives

$$|h(g(x)) - y_0| \leq \int_{x_0}^{x} M dt = M(x - x_0). \tag{21}$$

Next, recall that h is constrained by $h \in (0, \leq \min\left\{\alpha, \frac{\alpha}{M+K\alpha}\right\}]$, with $\alpha = \min\{a, b\}$, in the statement of the existence and uniqueness theorem. Note that this implies

$$h \leq \frac{\alpha}{M + K\alpha} = \frac{1}{\frac{M}{\alpha} + K} \leq \frac{1}{\frac{M}{b} + K} = \frac{b}{M + Kb} \leq \frac{b}{M}.$$

This means that since $x - x_0 \leq h \leq \frac{b}{M}$, plugging into equation 21

$$|h(g(x)) - y_0| \leq Mh \leq M\frac{b}{M} = b,$$

which proves the that if $|g(x) - y_0| \leq b$, then $|h(g(x)) - y_0| \leq b$ for $x \geq x_0$, with the same (unproved) result holding true for $x < x_0$. □

Stepping back, this means that for $x \in B_h[x_0]$ (the domain of g), if $g(x) \in B_b[y_0]$, we have that $h(g(x)) \in B_b[y_0]$, or that $(x, h(g(x))) \in R$. We defined $h \circ g$ such that $h(g(\Phi_n)) = \Phi_{n+1}$ in the Picard iteration. Thus, each term in the sequence has its graph in R as long as the previous term does; note that the graph of the first term $\Phi_0(x) = y_0$ is obviously in R, so by induction all terms also have their graphs in R.

5 Uniform Convergence [3]

Having shown that the sequence remains within the constraints of the problem, we would now like to show that it converges to the true solution. Suppose we knew the limit function, Φ, of this sequence. Then we would take the limit of the recursive formula:

$$\begin{aligned} \lim_{n\to\infty} [\Phi_n(x)] &= \lim_{n\to\infty} \left[y_0 + \int_{x_0}^{x} F(t, \Phi_{n-1}(t))dt \right] \\ \Phi(x) &= y_0 + \lim_{n\to\infty} \left[\int_{x_0}^{x} F(t, \Phi_{n-1}(t))dt \right]. \end{aligned}$$

Recall that the integral equation that we are trying to solve is

$$\Phi(x) = y_0 + \int_{x_0}^{x} F(t, \Phi(t))dt.$$

Thus, if we could swap the order of the limit and the integral, we would have shown that the limit function Φ solves the integral equation. Swapping those operations does not necessarily behave properly (see Buck, Chapter 6 if you'd like examples of when they fail [3]). To obtain conditions where they do, we will introduce uniformly convergent sequences, and show that Picard iteration produces such a sequence; additionally, this property allows us to show the continuity of the solution. [2]

Definition. *A sequence of functions* $f_1, ..., f_n$ *will be notated* $\{f_n\}$.

Definition. *A sequence* $\{f_n\}$ *is called* ***uniformly convergent*** *to a function* F *on set* E *if*

$$\lim_{n\to\infty} \|F - f_n\|_E = 0.$$

Intuitively, we might consider the convergence of functions to occur when $\lim_{n\to\infty} f_n(p) \to F(p)$ for every p; this is known as **pointwise convergence**. However, uniform convergence accounts for the entire function as a whole, instead of only individual points, and is stronger and more well-behaved in manners that we will not detail here (for examples, read the introduction to Buck Chapter 6 [3]). Spelled out, uniform convergence means that for any $\epsilon > 0$, there exists some N such that for any $n \geq N$, $|F(p) - f_n(p)| \leq \epsilon$ for all $p \in E$. By inspection, it should become evident that *uniform convergence implies pointwise convergence.* Now, with uniform convergence defined, we can show that a uniformly converging sequence of continuous functions has a continuous limit function.

Theorem 3. *If $\{f_n\}$ converges to F uniformly on E and each function f_n is continuous on E, then F is continuous on E.*

Proof. Suppose $\{f_n\}$ converges to F uniformly on E, with each f_n being continuous on E. We want to prove the continuity of F, which means that it must be continuous at each point $p_0 \in E$. By the epsilon-delta definition of continuity, this means that for each $p_0 \in E$ and $\epsilon > 0$, there must be some $\delta > 0$ such that for all $p \in E$ such that if $|p - p_0| < \delta$, then $|F(p) - F(p_0)| < \epsilon$.

For any $\epsilon > 0$, consider that since $\{f_n\}$ is uniformly convergent, there must exist some N such that $\|F - f_N\|_E < \epsilon/3$. Also, since f_N is continuous there exists some $\delta > 0$ such that if $|p - p_0| < \delta$, we must have $|f_N(p) - f_N(p_0)| < \epsilon/3$. We can introduce these terms into the expression we need to use to show the continuity of F, $|F(p) - F(p_0)|$, through the oft-used method of adding and subtracting the same quantities and applying the triangle method:

$$\begin{aligned} |F(p) - F(p_0)| &= |F(p) - f_N(p) + f_N(p) - f_N(p_0) + f_N(p_0) - F(p_0)| \\ &\leq |F(p) - f_N(p)| + |f_N(p) - f_N(p_0)| + |f_N(p_0) - F(p_0)|. \end{aligned}$$

Note that since $p, p_0 \in E$, we must have

$$|F(p) - f_N(p)|, |f_N(p_0) - F(p_0)| \leq \|F - f_N\|_E < \epsilon/3$$

We can substitute that in to obtain

$$|F(p) - F(p_0)| \leq 2\|F - f_N\|_E + |f_N(p) - f_N(p_0)| < \frac{2}{3}\epsilon + |f_N(p) - f_N(p_0)|.$$

Finally, recall that we stated that by the continuity of f_N, we selected p such that $|p - p_0| < \delta$, which implied that $|f_N(p) - f_N(p_0)| < \epsilon/3$. This can finally be plugged in to yield

$$|F(p) - F(p_0)| < \frac{2}{3}\epsilon + |f_N(p) - f_N(p_0)| < \frac{2}{3}\epsilon + \frac{1}{3}\epsilon = \epsilon.$$

Thus, we have shown that given $\epsilon > 0$ we can select a $\delta > 0$ such that if $|p - p_0|$, $|F(p) - F(p_0)| < \epsilon$, which satisfies the definition of continuity; since this is true for all $p_0 \in E$, we have proven that $\{f_n\}$ consists of continuous functions and converges uniformly to F, then F if continuous. ☐

5.1 Uniformly Cauchy Sequences [3] [4]

We will also briefly introduce uniformly Cauchy sequences and their relationship to uniformly convergent sequences.

Definition. *A sequence of functions $\{f_n\}$ is said to have the **Cauchy property uniformly**, or to be a **uniformly Cauchy sequence**, on a set E if for any $\epsilon > 0$, there exists an N such that for all $n, m \geq N$,*

$$\|f_n - f_m\|_E < \epsilon.$$

This is also written as

$$\lim_{n,m\to\infty} \|f_n - f_m\|_E = 0.$$

Theorem 4. *A sequence is uniformly convergent if and only if it has the Cauchy property uniformly.*

Proof. ($\Rightarrow$) Suppose that a sequence of real-valued functions $\{f_n\}$ defined on a set E is uniformly convergent, converging to F. Then for each $\epsilon/2 > 0$, there exists some N such that for $n \geq N$, for each $x \in E$ the following inequality holds:

$$|f_n(x) - F(x)| < \epsilon/2 \tag{22}$$

Now, also let $m \geq N$, so that inequality 22 holds as well. Then we can see by the triangle inequality that

$$|f_n(x) - f_m(x)| \leq |f_n(x) - F(x)| + |F(x) - f_m(x)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since this holds true for all $x \in E$, this means that

$$\|f_n - f_m\|_E < \epsilon,$$

so $\{f_n\}$ is uniformly Cauchy.

($\Leftarrow$) Suppose that a sequence of real-valued functions $\{f_n\}$ defined on a set E is uniformly Cauchy. Then for each $\epsilon/2 > 0$, there exists some N such that for any $n > m \geq N$ and $x \in E$,

$$|f_n(x) - f_m(x)| < \epsilon/2. \tag{23}$$

Since $n > m$, we can express it as $n = m + k$ for some $k \in \mathbb{Z}^+$. To match the convergence expression $|F(x) - f_m(x)|$, we take the limit $k \to \infty$, yielding

$$|F(x) - f_m(x)| = \lim_{k\to\infty} |f_{m+k}(x) - f_m(x)|$$

By the uniform Cauchy property, $|f_{m+k}(x) - f_m(x)| < \epsilon/2$, so we get

$$|F(x) - f_m(x)| = \lim_{k\to\infty} |f_{m+k}(x) - f_m(x)| \leq \epsilon/2 < \epsilon.$$

Since this is true for all x, this means that for each $\epsilon > 0$, there exists some N such that for all $m \geq N$,

$$\|F - f_m\|_E < \epsilon,$$

so if $\{f_n\}$ is uniformly Cauchy, it is uniformly convergent. □

This should make sense intuitively - as f_n gets closer and closer to the limit function F as $n \to \infty$, it should also be getting closer to other functions in the sequence f_m with large m. The use of this property comes from the fact that the Picard iteration is a recursively defined sequence, so we can relate f_n and f_m in a straightforward manner. We can then show the uniform Cauchy property - and thus uniform convergence - without even knowing the limit function!

Recall that the limit of the Picard iteration sequence is what we're trying to find - once we show that it's uniformly Cauchy, that means it's uniformly convergent. We'll then show that that property means we can take swap the order of the limit and the integral - our roadblock in the first place - and finally show that the limit of the Picard iteration is a solution, again still without necessarily knowing the solution.

6 Uniform Convergence of Picard Iteration [5]

We're now going to apply these uniform convergence results to Picard iteration. We'll restate the IVP for the sake of freshness:

$$\begin{aligned} y'(x) &= F(x, y(x)) \\ y(x_0) &= y_0, \end{aligned}$$

where F defined on the rectangle $R = B_a[x_0] \times B_b[y_0]$, with F and $\frac{\partial F}{\partial y}$ both continuous on R. We are trying to guarantee the existence of a local solution on $B_h[x_0]$, where $\alpha = \min\{a, b\}$ and $h \leq \min\left\{\alpha, \frac{\alpha}{M+K\alpha}\right\}$.
As previously mentioned, we'll start by trying to prove that this sequence is uniformly Cauchy.

Lemma 5. *The Picard iteration sequence* $\{\Phi_n\}$ *is uniformly Cauchy.*

Proof. Showing that a sequence is Cauchy involves the differences between iterates, which can be analyzed by substituting in their iterative definitions:

$$\begin{aligned} |\Phi_n(x) - \Phi_m(x)| &= \left| \int_{x_0}^{x} F(t, \Phi_{n-1}(t)) - F(t, \Phi_{m-1}(t)) dt \right| \\ &\leq \int_{x_0}^{x} |F(t, \Phi_{n-1}(t)) - F(t, \Phi_{m-1}(t))| dt. \end{aligned}$$

By again using the Mean Value Theorem and bounding the derivative,

$$|\Phi_n(x) - \Phi_m(x)| \leq K \|\Phi_{n-1} - \Phi_{m-1}\|_{B_h[x_0]} |x - x_0|$$

Using $x \in B_h[x_0]$ and our bound on h, we know that $|x - x_0| \leq h \leq \frac{\alpha}{M+K\alpha}$.

$$|\Phi_n(x) - \Phi_m(x)| \leq \frac{K\alpha}{M + K\alpha} \|\Phi_{n-1} - \Phi_{m-1}\|_{B_h[x_0]}.$$

We will define $C = \frac{K\alpha}{M+K\alpha}$; note that $C < 1$. Now, without loss of generality (due to the symmetry of absolute values and norms), we can suppose $n \geq m$. This process of substituting in the integral definitions can be iteratively repeated to show that

$$|\Phi_n(x) - \Phi_m(x)| \leq C^m \|\Phi_{n-m} - \Phi_0\|_{B_h[x_0]}.$$

Note that the right-hand side is independent of x, so it serves as a least upper bound and we can substitute in $\|\Phi_n - \Phi_k\|_{B_h[x_0]}$ on the left-hand side. Also, we can substitute in $\Phi_0(x) = y_0$ for all x.

$$\|\Phi_n - \Phi_m\|_{B_h[x_0]} \leq C^m \|\Phi_{n-k} - y_0\|_{B_h[x_0]}$$

Finally, since the values of all Φ_{n-k} must stay in $B_b[y_0]$, $\|\Phi_{n-k} - y_0\|_{B_h[x_0]} \leq b$. Thus,

$$\|\Phi_n - \Phi_m\|_{B_h[x_0]} \leq C^m b.$$

Observe that since $C < 1$, as $\lim_{m\to\infty} C^m b = 0$, so the left-hand side converges to 0 as $m \to \infty$. Since we said $n \geq m$, this means that $\lim_{n,m\to\infty} \|\Phi_n - \Phi_m\|_{B_h[x_0]} = 0$, so $\{\Phi_n\}$ is a uniformly Cauchy sequence. □

6.1 Showing that the Limit Function is a Solution to the Integral Equation [6]

In Lemma 5, we showed that $\{\Phi_n\}$ is uniformly Cauchy. Thus, by Theorem 4, $\{\Phi_n\}$ uniformly converges to some limit function Φ. As noted for motivation earlier, we'll show the limit $\{\Phi_n\}$ in fact gives a solution to the integral equation by simply taking said limits on the recursive definition:

$$\begin{aligned}\lim_{n\to\infty}[\Phi_n(x)] &= \lim_{n\to\infty}\left[y_0 + \int_{x_0}^{x} F(t,\Phi_{n-1}(t))dt\right]\\ \Phi(x) &= y_0 + \lim_{n\to\infty}\left[\int_{x_0}^{x} F(t,\Phi_{n-1}(t))dt\right].\end{aligned}$$

Earlier, our roadblock was in taking the limit into the integral sign. We will now show that this is allowed for uniformly convergent sequences of continuous functions.

Lemma 6. *Let $\{f_n\}$ be a sequence of continuous functions and converge uniformly on a closed interval $I = [a,b]$. Then*

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx.$$

Proof. Suppose $f_n(x)$ is comprised of continuous functions and uniformly convergent. Then it converges uniformly to some function f, and pointwise convergence holds:

$$\lim_{n\to\infty} f_n(x) = f(x).$$

We thus want to show that

$$\begin{aligned}\lim_{n\to\infty}\int_a^b f_n(x)dx &= \int_a^b f(x)dx\\ \lim_{n\to\infty}\int_a^b f_n(x) - f(x)dx &= 0.\end{aligned}$$

Alternatively, this means that for every $\epsilon > 0$, we must be able to find some N such that for all $n \geq N$, we have $\left|\int_a^b f_n(x) - f(x)dx\right| < \epsilon$.

Since $\{f_n\}$ is uniformly convergent, for each $\frac{\epsilon}{b-a} > 0$ there exists some N such that for all $n \geq N$, we have $\|f_n - f\|_I < \frac{\epsilon}{b-a}$. Since $|f_n(x) - f(x)| \leq \|f_n - f\|_I$ for all $x \in I$, we have

$$\begin{aligned}\left|\int_a^b f_n(x) - f(x)dx\right| &\leq \int_a^b |f_n(x) - f(x)|dx \leq \int_a^b \|f_n - f\|_I dx\\ &< \int_a^b \frac{\epsilon}{b-a}dx = \frac{\epsilon}{b-a}(b-a) = \epsilon.\end{aligned}$$

Thus, we have shown that

$$\lim_{n\to\infty}\int_a^b f_n(x) - f(x)dx = 0,$$

so

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx$$

for a uniformly converging sequence of continuous functions $\{f_n\}$. $\square$

With Lemma 6, we can now complete our proof, since $\{\Phi_n\}$ is a converging sequence of continuous functions:

$$\begin{aligned}
\lim_{n\to\infty}[\Phi_n(x)] &= \lim_{n\to\infty}\left[y_0 + \int_{x_0}^x F(t,\Phi_{n-1}(t))dt\right] \\
\Phi(x) &= y_0 + \lim_{n\to\infty}\left[\int_{x_0}^x F(t,\Phi_{n-1}(t))dt\right] \\
\Phi(x) &= y_0 + \int_{x_0}^x \lim_{n\to\infty} F(t,\Phi_{n-1}(t))dt \\
\Phi(x) &= y_0 + \int_{x_0}^x F(t,\Phi(t))dt.
\end{aligned}$$

This is exactly the integral equation, so the limit of the Picard iteration Φ (which is continuous by Theorem 3) constructively proves the existence of a local solution integral equation, and thus to the posed IVP. Additionally, note that by the Fundamental Theorem of Calculus, we can differentiate the integral equation with respect to x and obtain

$$\begin{aligned}
\frac{d}{dx}\Phi(x) &= \frac{d}{dx}\left[y_0 + \int_{x_0}^x F(t,\Phi(t))dt\right] \\
\Phi'(x) &= F(x,\Phi(x)).
\end{aligned}$$

Since F is continuous and composed of continuous functions, $F(x,\Phi(x))$ is continuous; thus, $\Phi'(x)$ is continuous, so the solution Φ to the IVP is $\mathcal{C}^1$. $\square$

6.2 Picard Iteration of the Kinetics Example

Recall the IVP from our kinetics example,

$$\begin{aligned}
A'(t) &= -0.2A(t) = F(t,A(t)) \\
A(0) &= 1,
\end{aligned}$$

which has the equivalent integral equation

$$A(t) = 1 + \int_0^t -0.2A(t)dt.$$

Let's use Picard iteration to find the unique solution to our problem on $R_1 = B_1[0] \times B_{e^{0.2}}[1]$ (which, hint hint, should match our intuitively found solution $A(t) = e^{-0.2t}$).

We construct first the sequence of approximations

$$\Phi_0(t) = 1$$
$$\Phi_1(t) = 1 + \int_0^t -0.2(1)dt = 1 + (-0.2)t$$
$$\Phi_2(t) = 1 + \int_0^t -0.2(1 - 0.2t)dt = 1 + (-0.2)t + \frac{(-0.2)^2}{2}t^2$$
$$\Phi_3(t) = 1 + \int_0^t -0.2(1 + (-0.2)t + \frac{(-0.2)^2}{2}t^2)dt = 1 + (-0.2)t + \frac{(-0.2)^2}{2}t^2 + \frac{(-0.2)^3}{3!}t^3.$$
$$\begin{aligned}\Phi_4(t) &= 1 + \int_0^t -0.2(1 + (-0.2)t + \frac{(-0.2)^2}{2}t^2 + \frac{(-0.2)^3}{3!}t^3)dt \\ &= 1 + (-0.2)t + \frac{(-0.2)^2}{2}t^2 + \frac{(-0.2)^3}{3!}t^3 + \frac{(-0.2)^4}{4!}t^4.\end{aligned}$$

With the n^{th} iteration, we seem to be adding on a term of the form $\frac{(-0.2)^n}{n!}t^n$, giving a general form for the n^{th} term:

$$\Phi_n(t) = \sum_{j=0}^{n} \frac{(-0.2)^j}{j!}t^j.$$

We can inductively prove that this is true. We can see plenty of base cases from our worked-out terms. Now suppose this is true for the n^{th} term. Then the $n+1^{th}$ term is

$$\begin{aligned}\Phi_{n+1}(t) &= 1 + \int_0^t -0.2\sum_{j=0}^{n} \frac{(-0.2)^j}{j!}t^j dt = 1 + \int_0^t \sum_{j=0}^{n} \frac{(-0.2)^{j+1}}{j!}t^j dt \\ &= 1 + \sum_{j=0}^{n} \frac{(-0.2)^{j+1}}{(j+1)!}t^{j+1} = \sum_{k=0}^{n+1} \frac{(-0.2)^k}{k!}t^k,\end{aligned}$$

which follows the sequence.

Now note that each of these series finite series $\Phi_n(t)$ is the n^{th} order Taylor polynomial for $e^{-0.2t}$. Thus, we can conclude that our unique solution must be

$$\Phi(t) = \lim_{n\to\infty} \Phi_n(t) = \lim_{n\to\infty} \sum_{j=0}^{n} \frac{(-0.2)^j}{j!}t^j = \sum_{j=0}^{\infty} \frac{(-0.2)^j}{j!}t^j = e^{-0.2t}$$

Thus, $\Phi(t) = e^{-0.2t}$ is the unique solution to the initial value problem,

$$\begin{aligned}&A'(t) = -0.2A(t) = F(t, A(t)) \\ &A(0) = 1,\end{aligned}$$

for $t \in B_1[0]$, just as we found on our own.

Note that the domain of our result is fairly limited, when it doesn't seem like there was a reason to limit ourselves to the rectangle $R_1 = B_1[0] \times B_{e^{0.2}}[1]$ other than to fit into our existence and uniqueness theorem. It is true that this is far from the largest rectangle that existence and uniqueness holds; this form of the theorem provides only a non-ideal local guarantee, which is likely not the goal in most situations (though still useful for many, since we are often studying a specific bounded space in real world applications). There are other forms of the existence and uniqueness theorem that can provide a global guarantee - this form was just chosen for its informative and straightforward proof.

7 Conclusion

We have now proven an existence and uniqueness theorem. That is, we have shown that there exists a unique, $\mathcal{C}^1$ local solution to the IVP

$$\begin{aligned} y'(x) &= F(x, y(x)) \\ y(x_0) &= y_0, \end{aligned}$$

where F is continuous and defined on a closed rectangle R and $\frac{\partial F}{\partial y}$ is continuous on R.
In order to prove this theorem, we did the following:

1. Converted the IVP into an equivalent integral equation through straightforward integration and differentiation. We used the integral equation extensively for our proofs. The ensuing results shown were thus applicable to the IVP by that equivalence.
2. Proved the uniqueness of a solution. We followed the style of injectivity proofs from the past, taking two hypothetical solutions to the problem and showing that they were the same.
3. Introduced Picard iteration, whose limit was a potential way to find a solution to the integral equation.
   (a) Showed that the limit of Picard iteration stayed within the bounds of our problem.
   (b) Introduced ideas about uniform convergence and uniformly Cauchy sequences.
   (c) Applied these ideas to the Picard iteration. Namely, we showed that it was uniformly Cauchy, and thus uniformly convergent. This allowed us to evaluate the limit of the integral equation, and thus show that the limit function of the Picard iteration was in fact a solution.
   (d) Showed that the solution was $\mathcal{C}^1$ by the Fundamental Theorem of Calculus.
4. Having shown the uniqueness and existence of a solution, we concluded that we had proven that there was a $\mathcal{C}^1$ solution to the given IVP.

Obviously, this problem - a single scalar ordinary differential equation defined on a closed and bounded region - represents a small portion of all of the possible IVPs involving differential equations. It can still be relevant to many models, though, since we often consider specific closed regions. Under different constraints, existence and uniqueness theorems have been proven for more general situations, including systems of equations and open domains. I hope you found this paper interesting enough to learn further about this topic, and thank you for reading.

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