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Section 2: Conditional Probability

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Based section template and practice problems by Rachel Li and Ginnie Ma '23
Practice problem selection also aided by Liz Li and Karina Wang

Forms

- Attendance form: http://bit.ly/110attend
- Feedback form: https://bit.ly/SrihariFeedback

Attendance form

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1 Summary

Notation 1. See that we use commas and intersections interchangeably (i.e., $P(A, B, C) = P(A \cap B \cap C)$).

Remark 2. A rough workflow for solving probability problems:

1. Define events for every aspect of the problem (e.g., "A = the event that it rains tomorrow, B = the event that it rained today")
2. Write out the probabilities that you are given in the problem using notation (e.g., "$P(A|B) = 1/2$, $P(B) = 1/4$).
3. Write the probability that you want to calculate using notation (e.g., we want to calculate the unconditional probability that it rains tomorrow, $P(A)$).
4. Figure out how the tools we have learned allow you to utilize the probabiliies that you do know (step 2) to calculate the probabilities that you don't know (step 3).

1.1 Definition of Probability

Definition 3 (Axioms of Probability)**.** With sample space S,

1. $P(S) = 1, P(\emptyset) = 0$.
2. For $A_1, A_2, \ldots,$ that partition B (this can be finite or infinite),

$$P(B) = \sum_{j=1}^{\infty} P(A_j)$$

Result 4 (Probability of a complement)**.** For event A,

$$P(A) = 1 - P(A^c)$$

Result 5 (Probability of a union)**.** For events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= P(A \cap B^c) + P(B \cap A^c) + P(A \cap B)$$

Result 6 (Principle of Inclusion-Exclusion)**.** For events $A_1, \ldots, A_n$,

$$P(\bigcup_{i=1}^{n} A_i) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

$$+ \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P(\bigcap_{i=1}^{n} A_i).$$

1.2 Conditional Probability

Definition 7 (Conditional probability)**.** The probability of event A given that we know B occured is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

...with extra conditioning:

$$P(A|B, C) = \frac{P(A \cap B|C)}{P(B|C)}.$$

1.3 Conditional Probability Tools

Remark 8 (First-step analysis)**.** If you ever need to solve a problem involving a sequence of things (like a game with many turns, or a random walk, or so on) and are stuck, try first-step analysis: conditioning what happens after the first step. You'll often be able to get a recursive equation that is easier to solve.

Result 9 (Probability of an intersection)**.**

$$P(A_1, A_2, \dots A_n) = P(A_1)P(A_2|A_1)\cdots P(A_n|A_1, \dots, A_{n-1})$$

$$= P(A_n)P(A_{n-1}|A_n)\cdots P(A_1|A_2, \dots, A_n),$$

$$= [\text{chaining in any order that is convenient for you}].$$

...with extra conditioning:

$$P(A_1, A_2, \dots A_n|C) = P(A_1|C)P(A_2|A_1, C)\cdots P(A_n|A_1, \dots, A_{n-1}, C)$$

Result 10 (Law of Total Probability (LOTP))**.** for events $A_1, A_2, \dots, A_n$ that partition S, we can find $P(B)$ by

$$P(B) = P(B, A_1) + P(B, A_2) + \cdots + P(B, A_n)$$

$$= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_n)P(A_n).$$

We pick $A_1, A_2, \dots, A_n$ to "condition on what we wish we knew." These are situations where you don't know $P(B)$, but you know $P(B|A_1)$, $(B|A_2)$, etc.

...with extra conditioning:

$$P(B|C) = P(B, A_1|C) + P(B, A_2|C) + \cdots + P(B, A_n|C)$$

$$= P(B|A_1, C)P(A_1|C) + P(B|A_2, C)P(A_2|C) + \cdots + P(B|A_n, C)P(A_n|C).$$

Result 11 (Bayes' Rule)**.** for events A, B, if we want to calculate $P(B|A)$ but can only know how to calculate $P(A|B)$,

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

$$= \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)},$$

where we commonly expand the denominator using the Law of Total Probability (LOTP).

...with extra conditioning:

$$P(B|A, C) = \frac{P(A|B, C)P(B|C)}{P(A|C)}$$

$$= \frac{P(A|B, C)P(B|C)}{P(A|B, C)P(B|C) + P(A|B^c, C)P(B^c|C)}$$

1.4 Independence

Definition 12 (Independence)**.** A, B are defined to be independent if

$$P(A, B) = P(A)P(B).$$

Note that if A, B are independent, then A^c, B^c are independent, as are A, B^c, and so on; in generality, for functions f, g, the events $f(A), g(B)$ are independent.

A set of events $A_1, A_2, \ldots, A_n$ is independent if any subset of the events $A_{j_1}, \ldots, A_{j_k}$ follows the equation.

$$P(A_{j_1}, \ldots, A_{j_k}) = P(A_{j_1}) \cdots P(A_{j_k}).$$

Basically, for any combination of independent events, we should be able to factor out the probabilities.

Definition 13 (Conditional independence)**.** A, B are conditionally independent given C if

$$P(A, B|C) = P(A|C)P(B|C).$$

VERY IMPORTANT:

- Disjointness and independence are not the same: in fact, if A, B are disjoint, then they are **VERY DEPENDENT**, because you know if A happens, then B definitely didn't happen!
- Independence and conditional independence are not the same/do not imply each other. There is no guarantee that independent events are conditionally independent, or vice versa.

2 Practice Problems

Warning for this week: these are some time-consuming problems! I do not expect you to be able to solve all of these in the allotted section time!

1. **[Probability results, maybe some LOTP]** For each of the following, fill in the blank with a $\geq, \leq, =$, or ?. You can reason mathematically or with a picture (e.g., Venn diagram)

$$P(B^c) \_\_\_ P(A)$$
$$P(A \cup B) \_\_\_ 1 - P(A^c \cap B^c)$$
$$P(A) \_\_\_ P(A \cap B^c).$$

2. **[LOTP]** You have a project due with two intermediate milestones (i.e., checkpoints that help you tell whether you're on track to complete on time). Let A_1 be the event that you complete your first milestone on time, A_2 be the event that you complete your second milestone on time, and A_3 be the event that you complete your project on time. For $j = 1, 2$,

$$P(A_{j+1}|A_j) = 0.8$$
$$P(A_{j+1}|A_j^c) = 0.3.$$

Also assume that if we know your status on the second milestone (whether you completed it on time or not), the first milestone is no longer relevant to whether you complete the project on time.

(a) State the previous paragraph in terms of independence or conditional independence.

(b) Find the probability that you complete the project on time, given that you complete the first milestone on time.

3. **[Bayes' Rule]** It's that time of year for Datamatch! A recent survey states that if a participant likes their match, there is a $\frac{3}{4}$ chance they will match back, and if they don't like their match, there is a $\frac{1}{2}$ chance they will match back (for the fooood). Let's be honest, you're a stunner so you assign a prior that your match likes you of $\frac{2}{3}$. What's the probability that your match likes you given that they matched back?

4. **[LOTP/first-step-analysis][If you have extra time!]** Two players (A, B) take turns tossing a fair coin, with A going first. The sequence of heads and tails is recorded, with H representing heads and T representing tails. If a head is followed by a tail, the player who flipped the tail wins. What is the probability A wins?