Wednesday, September 27 **Fall 2023**

Section 3: Conditional Probability Examples and Random Variables

Srihari Ganesh

Based on section note formatting template and a practice problem by Rachel Li and Ginnie Ma '23

Forms

- Attendance form: http://bit.ly/110attend
- Feedback form: https://bit.ly/SrihariFeedback

Attendance form

Section feedback form

1 Summary

1.1 Examples from class

Some takeaways:

- Winter girl: define events very specifically and see if you can simplify your problems with logic, not just relying on grinding through math
- Use the Law of Total Probability to condition on anything/everything you wish you knew
- In Monty Hall, condition on the location of the car!
- To mimic Simpson's paradox, set up some "hard" and "easy" tasks and make the better doctor do more of the hard tasks
- Learn how to turn a problem into the gambler's ruin problem:
  - Make losing happen at 0, and winning happen at some fixed N
  - Make sure each bet/step is only one dollar in either direction
  - Make sure the probability of winning each individual bet is constant

1.2 Random variables

Definition 1. Random variables are a numerical (real number) summary of the outcome of your experiment.

So for each possible outcome, the random variable takes on a certain value. Multiple outcomes can lead to the same value of the random variable.

Definition 2. The **support** of a random variable is the set of possible values it can take on.

Definition 3. We define discrete random variables by their **probability mass function**. You should

- define the probability $P(X = x)$ for each x in the random variable's support
- always write probability 0 for any value of x that is not in the support
- The PMF should always give valid probabilities, and should sum to 1 over all possible values

1.3 Distributions

Story 4 (Bernoulli distribution)**.** You conduct a single trial that succeeds with probability p. Let $X = 1$ if the trial succeeds and $X = 0$ if it fails. Then X follows a **Bernoulli distribution** with probability p, notated $X \sim \text{Bern}(p)$, and has a PMF

$$P(X = x) = \begin{cases} p & x = 1 \\ 1 - p & x = 0 \\ 0 & \text{else} \end{cases}$$

Story 5 (Binomial distribution)**.** You conducts n independent trials that each succeed with probability p. Let Y be the total number of successes among the n trials. Then Y is distributed Binomial with n trials and probability p, notated $Y \sim \text{Bin}(n, p)$, and the PMF is

$$P(Y = y) = \begin{cases} \binom{n}{y} p^y (1 - p)^{n-y} & y \in \{0, 1, \ldots, n\} \\ 0 & \text{else} \end{cases}$$

2 Practice Problems

1. **Working Bernoulli and Binomial stories and distributions** (NOTE: these are all properties you should know moving forward)

   (a) Suppose $X \sim \text{Bern}(p)$. What is the distribution of $Y = 1 - X$? What about if $X \sim \text{Bin}(n, p)$ and $Y = n - X$?

   (b) Suppose $X_1, X_2, \ldots, X_m \sim \text{Bern}(1/m)$ independently. What is the distribution of $\sum_{i=1}^{m} X_i$?

   (c) Suppose $X_1 \sim \text{Bin}(n, p)$ and $X_2 \sim \text{Bin}(m, p)$ with X_1, X_2 independent. What is the distribution of $X_1 + X_2$?

   (d) Suppose $X \sim \text{Bern}(p)$. What is the distribution of X^2?

2. **[Working with distributions and probability calculations]**You go to an online store and see 5 one-of-a-kind scarves. For each scarf, you decide to roll a fair die and buy the scarf if you roll a 4. Each scarf costs $13 (I have now clue how much scarves cost). If you end up ordering anything, you will also have to pay a flat $8 shipping fee (i.e., it's $8 whether you get 2 scarves or 5). What's the probability you pay more than $5, but no more than $60?

3. **[Gambler's ruin]** I am pursuing a very unconventional degree program. Each semester, I take a single class that I have a probability p of failing. I am allowed to graduate if, at any point in time, I have passed 20 more classes than I have failed (up to that point). I get kicked out of school if, at any point, I have failed more classes than I have passed (up to that point). I get an unlimited amount of time to try to graduate as long as I haven't been kicked out yet. What's the probability that I eventually end up graduating?

4. **[Monty Hall: More Monty]** *(taken from Rachel & Ginnie's 2022 section notes)*

   There are three doors, behind one of which there is a car and behind the other two of which there are goats. You want the car (hopefully). Initially, from your perspective, all possibilities are equally likely for where the car is. You choose a door, which we'll say is Door 1. Monty Hall opens a door with a goat in it and then offers you the option of switching. In class, in the case that Monty Hall knows where the car is and would never reveal the car, the probability that the strategy of always switching succeeds is 2/3. Find the probability that the strategy of always switching succeeds if:

   (a) Monty does not know what lies behind the doors and opens Door 2 at random (50/50), which happens to not reveal the car.

   (b) Monty always reveals a goat (he knows where the car is) and, if he has a choice, he always reveals the leftmost goat (which may depend on the player's choice). In this case, he has opened Door 2 knowing you chose Door 1.